Kuwait University
Department of Mathematics

Math 101
Second Exam

July 23, 2011
Time: 90 min.

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**Calculators and Mobile Phones are not allowed.**

1. **[2 + 1 Points]** Let $f(x) = 1 + \sin x$.

   a) Use differentials to find expressions for $\Delta y$ and $dy$.

   b) Use the linear approximation formula to estimate $f(29°)$.

2. **[4 Points]** Find an equation of the normal line at $x = 0$ to the graph of

$$xy^2 + y\sin(x) + y^3 = 1.$$

3. **[4 Points]** The length $L$ of a rectangle is decreasing at the rate of 2 cm/sec while the width $W$ is increasing at the rate of 1 cm/sec. Find the rate of change of the area $A$ of the rectangle when $L = 12$ cm and $W = 5$ cm.

4. **[1 + 3 Points]**

   a) State the Mean Value Theorem.

   b) Use the Mean Value Theorem to show that if $f$ is differentiable on $\mathbb{R}$, $f(0) = 1$, and $f'(x) \geq 2, \forall x > 0$, then

$$f(x) \geq 2x + 1, \forall x > 0.$$

5. **[2 + 2 + 2 + 2 +2 Points]** Let $f(x) = \dfrac{x^2}{(x-2)^2}$, and given that $f'(x) = \dfrac{-4x}{(x-2)^3}$ and $f''(x) = \dfrac{8(x+1)}{(x-2)^4}$.

   a) Find the vertical and horizontal asymptotes for the graph of $f$, if any.

   b) Find the intervals on which the graph of $f$ is increasing and the intervals on which the graph of $f$ is decreasing. Find the local extrema of $f$, if any.

   c) Find the intervals on which the graph of $f$ is concave upward and the intervals on which the graph of $f$ is concave downward. Find the points of inflection, if any.

   d) Sketch the graph of $f$.

   e) Find the absolute maximum and minimum values of $f$ on [-1,1].

1/ (a) $\Delta y = f(x+\Delta x) - f(x) = [1+\sin(x+\Delta x)] - [1+\sin x] = \sin(x+\Delta x) - \sin x$

$dy = f'(x)dx = \cos x \, dx$

(b) $30^\circ = \frac{\pi}{6}$ rd. At $x = \frac{\pi}{6}$, $L(x) = f(\frac{\pi}{6}) + f'(\frac{\pi}{6})(x - \frac{\pi}{6}) = \frac{3}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6})$

$f(29^\circ) \simeq L(29^\circ) = L\left(\frac{29\pi}{180}\right) = \frac{3}{2} - \frac{\sqrt{3}\pi}{360}$.

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2/ $y^2 + 2xyy' + y'\sin x + y\cos x + 3y^2y' = 0 \Rightarrow y' = \frac{-y^2 - y\cos x}{2xy + \sin x + 3y^2}$

At $x=0, y=1 \Rightarrow y'\Big|_{x=0} = -\frac{2}{3} \Rightarrow$ eq. of normal line: $y - 1 = \frac{3}{2}(x-0)$

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3/ $A = L.W \Rightarrow \frac{dA}{dt} = \frac{dL}{dt}W + L\frac{dW}{dt} \Rightarrow \frac{dA}{dt}\Big|_{\substack{L=12\\W=5}} = (-2)(5)+(12)(1) = 2 \text{ cm}^2/\text{sec}$

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4/ a) Notes.

b) $f$ is diff on $(0,x)$ and cont. on $[0,x]$. So by MVT on $[0,x]$

$f(x) - f(0) = f'(c)(x-0)$ for some $c \in (0,x)$

$\Rightarrow f(x) = 1 + f'(c)x \geq 1 + 2x$ because $f'(c) \geq 2$.

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5/ (a) $\lim_{x\to\pm\infty} \frac{x^2}{(x-2)^2} = \lim_{x\to\pm\infty} \frac{1}{\left(1-\frac{2}{x}\right)^2} = 1 \Rightarrow y = 1$ is a H.A.

$\lim_{x\to 2^\pm} \frac{x^2}{(x-2)^2} = +\infty \Rightarrow x = 2$ is a V.A.

(b)

| $x$ | $-\infty$ | 0 | | 2 | $\infty$ |
|---|---|---|---|---|---|
| $x$ | $-$ | 0 | $+$ | | $+$ |
| $x-2$ | $-$ | | $-$ | | $+$ |
| $f'(x)$ | $-$ | 0 | $+$ | | $-$ |
| $f(x)$ | ↘ | | ↗ | | ↘ |

$f$ is increasing on $(0,2)$

$f$ is decreasing on $(-\infty,0)\cup(2,\infty)$

$f(0) = 0$ is a local min. value of $f$ (because $f\searrow$ before 0 & $f\nearrow$ after 0) c)

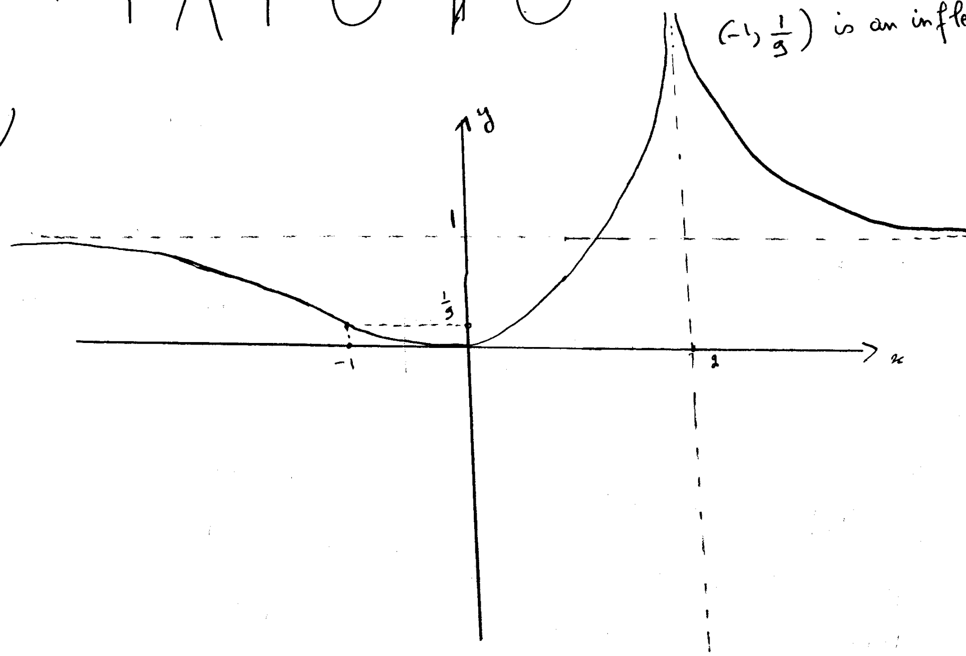
| $x$ | $-\infty$ | $-1$ | | $2$ | | $\infty$ |
|---|---|---|---|---|---|---|
| $x+1$ | $-$ | $0$ | $+$ | | $+$ | |
| $f''(x)$ | $-$ | $0$ | $+$ | | $+$ | |
| graph($f$) | $\cap$ | | $\cup$ | | $\cup$ | |

graph($f$) is concave up on $(-1,2)\cup(2,\infty)$

graph($f$) is concave down on $(-\infty,-1)$

$(-1,\frac{1}{9})$ is an inflection pt.

d)




e) $f$ has only one interior critical point, namely $x=0$. $f(0)=0$.

At the endpoints: $f(-1)=\frac{1}{9}$

$f(1)=1$

Therefore, the absolute max. of $f$ on $[-1,1]$ is $1$ attained at $x=1$.

the absolute min. of $f$ on $[-1,1]$ is $0$ attained at $x=0$.